

Density operators and selective measurements.

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1. Introduction.

It is widely believed that statistical interpretation of quantum mechanics requires that density operators representing quantum states be normalized. We present a description of selective measurements in terms of density operators. The description is inspired by Schwinger's Algebra of Microscopic Measurements [1], (see also [2]). Density operators used are not normalized. We do not know applications of density operators requiring normalization.

2. Beams of particles.

The physical space is an affine space M of dimension 3 modelled on a vector space V . There is an Euclidean metric tensor

$$g: V \rightarrow V^*. \quad (1)$$

We consider beams of particles of mass m and constant energy E in the direction of a unit vector $z \in V$. The internal states of the particles are elements of a unitary vector space U of dimension r over the field \mathbb{C} of complex numbers. The elements of U are *kets* $|u\rangle$ and elements of the dual space U^* are *bras* $\langle a|$. The unitary structure establishes an antilinear isomorphism of U with U^* assigning to each ket $|u\rangle$ a unique bra $\langle u^\dagger|$. The number

$$\langle u_1^\dagger | u_2 \rangle \quad (2)$$

is the *scalar product* of vectors $|u_1\rangle$ and $|u_2\rangle$

We introduce a sequence of points

$$x_0, x_1, x_2, \dots, x_n \quad (3)$$

satisfying inequalities

$$\langle g(z), x_i - x_{i-1} \rangle > 0, \quad (4)$$

and a corresponding sequence of planes

$$X_i = \{x \in M; \langle g(z), x - x_i \rangle = 0\}. \quad (5)$$

In the immediate neighbourhood of each plane X_i the beam is not subject to external interaction and is represented by a plane wave

$$|\psi_i(x)\rangle = |A_i\rangle \exp(ik\langle g(z), x - x_0 \rangle + \delta_i), \quad |A_i\rangle \in U, \quad k = \frac{\sqrt{2mE}}{\hbar} \quad (6)$$

with time dependence separated. The probability flux through a unit surface element of the plane X_i is expressed by

$$\frac{\hbar k}{m} \langle A_i^\dagger | A_i \rangle. \quad (7)$$

Between the plane X_{i-1} and the plane X_i the beam passes through a selective device. Its action on the wave function is described as the action of a linear transition operator

$$M_i : U \rightarrow U : |\psi_{i-1}(x)\rangle \mapsto |\psi_i(x)\rangle = M_i |\psi_{i-1}(x)\rangle. \quad (8)$$

If

$$M_1, M_2, M_3, \dots, M_n \quad (9)$$

is the sequence of transition operators and

$$|\psi_0(x)\rangle = |A_0\rangle \exp(ik\langle g(z), x - x_0 \rangle) \quad (10)$$

is the initial state, then

$$\begin{aligned} |\psi_i(x)\rangle &= |A_i\rangle \exp(ik\langle g(z), x - x_0 \rangle + \delta_i) \\ &= M_i \cdots M_2 M_1 |A_0\rangle \exp(ik\langle g(z), x - x_0 \rangle) \end{aligned} \quad (11)$$

and

$$\langle \psi_i^\dagger(x) | \psi_i(x) \rangle = \langle A_0^\dagger | M_1^\dagger M_2^\dagger \cdots M_i^\dagger M_i \cdots M_2 M_1 | A_0 \rangle \quad (12)$$

The flux of particles through unit surface element of X_i is given by

$$\frac{\hbar k}{m} \langle A_0^\dagger | M_1^\dagger M_2^\dagger \cdots M_i^\dagger M_i \cdots M_2 M_1 | A_0 \rangle. \quad (13)$$

3. Mixed states and density operators.

The expression (13) can be presented in the form

$$\frac{\hbar k}{m} \text{tr} (M_i \cdots M_2 M_1 | A_0 \rangle \langle A_0^\dagger | M_1^\dagger M_2^\dagger \cdots M_i^\dagger). \quad (14)$$

In this new expression the pure initial state is represented by the density operator $|A_0\rangle\langle A_0^\dagger|$ and can be replaced by a mixed state represented by a positive Hermitian density operator \mathbf{T} . The expression

$$\frac{\hbar k}{m} \text{tr} (M_i \cdots M_2 M_1 \mathbf{T} M_1^\dagger M_2^\dagger \cdots M_i^\dagger) \quad (15)$$

is the result.

4. Selective measurements.

We set the density operator \mathbf{T} in the expression (15) equal to

$$\mathbf{T} = \frac{m}{\hbar k r} \mathbf{I}, \quad (16)$$

where \mathbf{I} is the identity operator. The operator \mathbf{T} is normalized in the sense that

$$\frac{\hbar k}{m} \text{tr} \mathbf{T} = 1 \quad (17)$$

although this normalization is of no importance. The expression

$$P_i = \frac{\hbar k}{m} \text{tr} (M_i \cdots M_2 M_1 \mathbf{T} M_1^\dagger M_2^\dagger \cdots M_i^\dagger) = \frac{1}{r} \text{tr} (M_i \cdots M_2 M_1 M_1^\dagger M_2^\dagger \cdots M_i^\dagger) \quad (18)$$

represents the probability of detecting a particle crossing a unit surface element of the plane X_i in unit time. The state of the initial beam emitted by a source at X_0 is totally mixed.

We want to describe the following experimental arrangement. The initial beam emitted at X_0 undergoes a preliminary selection by a sequence of devices represented by the sequence of operators

$$M_1, M_2, \dots, M_j. \quad (19)$$

The beam undergoes further selection passing through a sequence of devices represented by operators

$$N_1 = M_{j+1}, N_2 = M_{j+2}, \dots, N_{n-j} = M_n. \quad (20)$$

The particles are detected at X_n by a non selective detector. After the preliminary selection the state of the beam is represented by the density operator

$$\mathbf{M} = M_j M_{j-1} \dots M_1 \mathbf{T} M_1^\dagger \dots M_{j-1}^\dagger M_j^\dagger = \frac{m}{\hbar k r} M_j M_{j-1} \dots M_1 M_1^\dagger \dots M_{j-1}^\dagger M_j^\dagger \quad (21)$$

with the probability of non selective detection

$$P_{in} = \frac{\hbar k}{m} \text{tr} (M_j M_{j-1} \dots M_1 \mathbf{T} M_1^\dagger \dots M_{j-1}^\dagger M_j^\dagger) = \frac{\hbar k}{m} \text{tr} \mathbf{M} \quad (22)$$

The beam arrives at X_n in a state represented by the density operator

$$N_{n-j} N_{n-j-1} \dots N_1 M_j M_{j-1} \dots M_1 \mathbf{T} M_1^\dagger \dots M_{j-1}^\dagger M_j^\dagger N_1^\dagger \dots N_{n-j-1}^\dagger N_{n-j}^\dagger \quad (23)$$

It is detected with the probability

$$\begin{aligned} P_{out} &= \frac{\hbar k}{m} \text{tr} (N_{n-j} N_{n-j-1} \dots N_1 M_j M_{j-1} \dots M_1 \mathbf{T} M_1^\dagger \dots M_{j-1}^\dagger M_j^\dagger N_1^\dagger \dots N_{n-j-1}^\dagger N_{n-j}^\dagger) \\ &= \frac{\hbar k}{m} \text{tr} (N_1^\dagger \dots N_{n-j-1}^\dagger N_{n-j}^\dagger N_{n-j} N_{n-j-1} \dots N_1 M_j M_{j-1} \dots M_1 \mathbf{T} M_1^\dagger \dots M_{j-1}^\dagger M_j^\dagger) \\ &= \frac{\hbar k}{m} \text{tr}(\mathbf{N} \mathbf{M}) \end{aligned} \quad (24)$$

with

$$\mathbf{N} = N_1^\dagger \dots N_{n-j-1}^\dagger N_{n-j}^\dagger N_{n-j} N_{n-j-1} \dots N_1. \quad (25)$$

The density operator \mathbf{N} characterizes the selective detector. The probability P_{out} is measured at X_j by the selective detector. This measurement is performed on the mixed state represented by the operator \mathbf{M} . The relative probability

$$P_{out}/P_{in} = \text{tr}(\mathbf{N} \mathbf{M})/\text{tr} \mathbf{M} \quad (26)$$

should be considered the result of the selective measurement described. Arbitrary normalization can be imposed on \mathbf{M} . Normalization of \mathbf{N} would distort the result of the measurement.

5. An example.

In addition to the metric tensor

$$g: V \rightarrow V^* \quad (27)$$

we introduce in the model space V of the physical space M an orientation o defined as an equivalence class of bases.

We analyse the internal states of a beam of particles of spin 1/2. States of particles are represented by wave functions with values in an unitary space of complex dimension 2. The set of hermitian traceless operators in U is a real vector space S of dimension 3.

The trace $\text{tr}(ab)$ of a product is a non negative real number and the mapping

$$S \times S \rightarrow \mathbb{R}: (a, b) \mapsto \text{tr}(ab) \quad (28)$$

is bilinear and symmetric. The spectrum of an operator $a \in S$ is a pair $\{\alpha, -\alpha\}$ of real numbers and the spectrum of the operator aa is the set $\{\alpha^2, \alpha^2\}$. It follows that $\text{tr}(aa) = 0$ if and only if $a = 0$. In conclusion we have a Euclidean *scalar product*

$$(\mid): S \times S \rightarrow \mathbb{R}: (a, b) \mapsto (a|b) = \frac{1}{2} \text{tr}(ab). \quad (29)$$

We introduce the *Pauli morphism*

$$\sigma: V \rightarrow S. \quad (30)$$

This morphism is an isometry such that the operator

$$\frac{1}{2}\sigma(w): U \rightarrow U \quad (31)$$

associated with each unit vector $w \in V$ is the *spin operator* in the direction w . Its spectrum is the set $\{1/2, -1/2\}$ and its eigenvectors represent states of the particle with spin $1/2$ and $-1/2$ in the direction of w .

We introduce a number of operators in the space U :

- 1) The projection operator

$$K(w) = \frac{1}{2}(I + \sigma(w)) \quad (32)$$

associated with a unit vector $w \in V$. This operator projects onto the space of eigenstates of the spin operator $1/2\sigma(w)$ corresponding to the eigenvalue $1/2$.

- 2) A *phase shift* operator

$$D(\delta) = \exp(i\delta)I. \quad (33)$$

- 3) An *attenuation* operator

$$R(\rho) = \exp(-\rho/2)I. \quad (34)$$

- 4) A unitary unimodular operator

$$G: U \rightarrow U. \quad (35)$$

This operator represents a rotation

$$E: V \rightarrow V \quad (36)$$

in the sense that

$$G\sigma(w)G^{-1} = \sigma(Ew). \quad (37)$$

- 5) The operator

$$Q(w) = \frac{1}{2}(I + \sigma(w)) \quad (38)$$

associated with a vector $w \in V$ of norm $\|w\| \neq 1$. This operator is not a projection operator.

Consider a beam undergo a preliminary selection by devices represented by $M_1 = K(w)$ and $M_2 = D(\delta)$. The vector w is orthogonal to the direction of the beam and the first of the devices is a Stern-Gerlach filter. It is accompanied by an unavoidable phase shift. The state prepared by these devices is a pure state represented by the density operator

$$\mathbf{M} = M_2 M_1 T M_1^\dagger M_2^\dagger = \frac{m}{2\hbar k} K(w). \quad (39)$$

The selective detector is composed of devices represented by operators $N_1 = R(\rho)$, $N_2 = D(\delta')$, and $N_3 = K(w')$. The attenuation $R(\rho)$ may be due to the beam passing through a potential barrier. The density operator

$$\mathbf{N} = N_1^\dagger N_2^\dagger N_3^\dagger N_3 N_2 N_1 = \exp(-\rho) K(w') \quad (40)$$

represents the selective detector. The result of the selective measurement is the relative probability

$$P_{out}/P_{in} = \text{tr}(\mathbf{N}\mathbf{M})/\text{tr} \mathbf{M} = \frac{1}{2} (1 + (w'|w)) \quad (41)$$

since

$$\text{tr} \mathbf{M} = 1 \quad (42)$$

and

$$\begin{aligned} \text{tr}(\mathbf{N}\mathbf{M}) &= \text{tr} (\exp(-\rho)K(w')K(w)) \\ &= \frac{1}{4} \text{tr} (\exp(-\rho)(I + \sigma(w'))(I + \sigma(w))) \\ &= \frac{1}{4} \text{tr} (\exp(-\rho)(I + \sigma(w') + \sigma(w) + \sigma(w')\sigma(w))) \\ &= \frac{1}{2} (1 + (w'|w)) \end{aligned} \quad (43)$$

6. References.

- [1] J. Schwinger, *The Algebra of Microscopic Measurements*, Proc. Natl. Acad. Sc. US, **45** (1959)
- [2] F. A. Kaempffer, *Concepts in Quantum Mechanics*, Academic Press, New York and London (1965)